

THE GELFAND-TSETLIN GRAPH AND MARKOV PROCESSES

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ABSTRACT. The goal of the paper is to describe new connections between representation theory and algebraic combinatorics on one side, and probability theory on the other side.

The central result is a construction, by essentially algebraic tools, of a family of Markov processes. The common state space of these processes is an infinite dimensional (but locally compact) space Ω . It arises in representation theory as the space of indecomposable characters of the infinite-dimensional unitary group $U(\infty)$.

Alternatively, Ω can be defined in combinatorial terms as the boundary of the Gelfand-Tsetlin graph — an infinite graded graph that encodes the classical branching rule for characters of the compact unitary groups $U(N)$.

We also discuss two other topics concerning the Gelfand-Tsetlin graph:

(1) Computation of the number of trapezoidal Gelfand-Tsetlin schemes (one could also say, the number of integral points in a truncated Gelfand-Tsetlin polytope). The formula we obtain is well suited for asymptotic analysis.

(2) A degeneration procedure relating the Gelfand-Tsetlin graph to the Young graph by means of a new combinatorial object, the Young bouquet.

At the end we discuss a few related works and further developments.

1. INTRODUCTION

The present paper is devoted to combinatorial and probabilistic aspects of the asymptotic representation theory. The adjective “asymptotic” means that we are interested in the limiting behavior of representations of growing groups

$$G(1) \subset G(2) \subset G(3) \subset \dots$$

and their relationship with representations of the limiting group $G(\infty)$, which is defined as the union of $G(N)$ ’s. Here there is a remarkable analogy with limit transitions in models of statistical physics and random matrix theory.

The model examples of the “big groups” $G(\infty)$ are the infinite symmetric group $S(\infty)$ and the infinite-dimensional unitary group $U(\infty)$. There is a striking parallelism between the theories for these two groups that we substantially exploit. In our exposition, we focus on the more difficult case of $U(\infty)$ and only briefly mention the parallel results concerning $S(\infty)$.

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The main references are the recent joint papers [11], [12], [13] by Alexei Borodin and myself, and my paper Olshanski [47]. These works originated from our previous study of the problem of harmonic analysis on $S(\infty)$ and $U(\infty)$ (Borodin-Olshanski [8], [9], [10]).

1.1. Relative dimension in the Gelfand-Tsetlin graph. All our considerations are intimately connected with the Gelfand-Tsetlin graph. We recall its definition in Section 3. As was already mentioned in the abstract, the Gelfand-Tsetlin graph encodes the branching of irreducible characters for the compact groups

$$U(1) \subset U(2) \subset U(3) \subset \dots \quad (1.1)$$

A fundamental result in the asymptotic representation theory is the Edrei-Voiculescu theorem on the classification of indecomposable characters of the group $U(\infty)$ (Edrei [19], Voiculescu [54]). In combinatorial terms, the Edrei-Voiculescu theorem describes the *boundary* of the Gelfand-Tsetlin graph (see Section 3 for the precise definitions).

In Borodin-Olshanski [11] we propose a novel approach to this old theorem, based on the study of the *relative dimension*

$$F_{\varkappa}(\nu) := \frac{\text{Dim}_{K,N}(\varkappa, \nu)}{\text{Dim}_N \nu}. \quad (1.2)$$

Here \varkappa and ν are two vertices of the Gelfand-Tsetlin graph, on levels K and N , respectively ($K < N$); the numerator is the number of monotone paths in the graph connecting \varkappa to ν ; finally, the denominator is the number of all monotone paths ending at ν (this is the same as the dimension of the irreducible character of $U(N)$ corresponding to ν). The notation in (1.2) emphasizes that we regard the ratio on the right-hand side as a function in variable ν with \varkappa being a fixed parameter.

We show that $F_{\varkappa}(\nu)$ is a rather “regular” function, which shares some properties of the classic Schur functions like the examples in Macdonald [35]. What is especially important for our purposes, we obtain a good contour integral representation for $F_{\varkappa}(\nu)$, which makes it possible to find its asymptotics as ν goes to infinity.

These results are reviewed in Section 3.

1.2. The zw-measures and related Markov processes. One of the most beautiful hypergeometric identities is classic Dougall’s formula (1907) which can be written as

$$\begin{aligned} & \sum_{n \in \mathbb{Z}} \frac{1}{\Gamma(z - n + 1)\Gamma(z' - n + 1)\Gamma(w + n + 1)\Gamma(w' + n + 1)} \\ &= \frac{\Gamma(z + w + z' + w' + 1)}{\Gamma(z + w + 1)\Gamma(z + w' + 1)\Gamma(z' + w + 1)\Gamma(z' + w' + 1)}, \end{aligned} \quad (1.3)$$

see Dougall [17], Erdelyi [20]. Here z, z', w, w' are complex parameters such that $\Re(z + z' + w + w') > -1$ and $\Gamma(\cdot)$ is Euler’s Γ -function. Let $M_{z,z',w,w'}(n)$ denote the n th summand on the left-hand side divided by the quantity on the right-hand side.

It is easy to find conditions under which all the summands on the left-hand side are real and positive. Then the quantities $M_{z,z',w,w'}(n)$, $n \in \mathbb{Z}$, define a probability measure $M_{z,z',w,w'}$ on \mathbb{Z} . We call it the *zw-measure*.

The zw-measures also arise in a probabilistic context. Recall that a *birth-death process* is a continuous time Markov chain (or random walk) on \mathbb{Z}_+ such that the only possible transitions from a state $n \in \mathbb{Z}_+$, in an infinitesimal time interval $(t, t + dt)$, are the neighboring states $n \pm 1$. Such a process is determined by specifying the jump rates $a_{\pm}(n)$; then the infinitesimal generator of the process is the difference operator $D_{z,z',w,w'}$ on \mathbb{Z}_+ acting on a test function f by

$$(Df)(n) = a_+(n)[f(n+1) - f(n)] + a_-(n)[f(n-1) - f(n)], \quad n \in \mathbb{Z}_+.$$

The birth-death processes are well-studied objects which have many applications.

Let us now ask what could be the simplest *bilateral* analog of birth-death processes, living on the whole lattice \mathbb{Z} and possessing a stationary distribution (in other words, an invariant probability measure). The generator of a bilateral process still has the same form, only the jump rates $a_{\pm}(n)$ are required to be strictly positive for all $n \in \mathbb{Z}$.

If the quantities $a_{\pm}(n)$ are constants, then there is no finite invariant measure. If $a_{\pm}(n)$ depends on n linearly, it changes the sign, which is inadmissible. But if we require $a_{\pm}(n)$ to be quadratic functions of n , then the processes with desired properties exist, they depend on four parameters, and the corresponding invariant measures are just the zw-measures.

Thus the zw-measures appear as the stationary distributions of certain natural Markov processes on \mathbb{Z} . Each of them is uniquely determined by its generator, which is the difference operator

$$\begin{aligned} (D_{z,z',w,w'}f)(n) &= (z-n)(z'-n)(f(n+1) - f(n)) \\ &\quad + (w+n)(w'+n)(f(n-1) - f(n)). \end{aligned} \quad (1.4)$$

Here n ranges over \mathbb{Z} and the parameters are subject to constraints stated in the beginning of Section 4 below.

Observe now that \mathbb{Z} is the Pontryagin dual group to the unit circle

$$\mathbb{T} := \{u \in \mathbb{C} : |u| = 1\},$$

which is a commutative group isomorphic to $U(1)$. In Sections 4–6 we explain how to construct analogs of the zw-measures, the related Markov processes, and the generators $D_{z,z',w,w'}$ when \mathbb{Z} is replaced by the *dual objects* to noncommutative groups $U(N)$ ($N = 2, 3, \dots$) and (which is our final goal) by the dual object to the group $U(\infty)$.

For $N = 2, 3, \dots$, the dual object $\widehat{U(N)}$, like $\widehat{U(1)} = \mathbb{Z}$, is a countable set; it is naturally identified with a subset $\mathbb{S}_N \subset \mathbb{Z}^N$ (the highest weights of the group $U(N)$). The dual object $\widehat{U(\infty)}$, on the contrary, is a continuous infinite-dimensional space:

its points depend on infinitely many continuous parameters. Thus our construction leads to a four-parameter family of Markov processes on this infinite-dimensional space.

The generators of these processes are explicitly computed: they are implemented by certain infinite-variate second order partial differential operators $\mathbb{D}_{z,z',w,w'}$ (see Section 6 below). Initially, $\mathbb{D}_{z,z',w,w'}$ is defined on a certain algebra R^U — the *representation ring* of the family $\{U(N); N = 1, 2, 3 \dots\}$. As is well known, the representation ring for the family of symmetric groups is isomorphic to Sym , the algebra of symmetric functions. We regard the algebra R^U as a reasonable substitute of the algebra Sym even though R^U seems to be more sophisticated as compared to Sym .

1.3. The Young bouquet. There exists a great similarity between the representation theories of the two basic big groups, $U(\infty)$ and $S(\infty)$. It is striking when comparing the description of the dual objects (cf. Voiculescu [54] and Thoma [51]) or the construction of the generalized regular representations which are the subject of harmonic analysis (cf. Olshanski [42] and Kerov-Olshanski-Vershik [27], [28]). The study of the relative dimension (1.2) in the Gelfand-Tsetlin graph has been inspired by earlier results (Okounkov-Olshanski [39]) on the relative dimension in the Young graph — the counterpart of the Gelfand-Tsetlin graph in the symmetric group case. The zw -measures and related Markov processes also have counterparts in the symmetric group case (Borodin-Olshanski [14]).

This parallelism is in sharp contrast to the fact that the groups $U(\infty)$ and $S(\infty)$, as well as $U(N)$ and $S(N)$, look quite differently. In Borodin-Olshanski [13] we suggest an explanation of this phenomenon. The idea is that one can establish a connection between the Gelfand-Tsetlin and Young graphs by making use of a certain poset with *continuous grading*. We call this poset the *Young bouquet*.

By the very definition, the Young bouquet is a close relative of the Young graph. On the other hand, we show that the Young bouquet can be obtained from the Gelfand-Tsetlin graph by a limit transition turning the discrete grading into a continuous one. Moreover, the limit transition leads to a reasonable degeneration of various objects that are structurally connected with the Gelfand-Tsetlin graph.

We discuss the Young bouquet in Section 7. Note that the results of [13] are substantially used in the construction of Markov processes in the symmetric group case (Borodin-Olshanski [14]).

2. DUAL OBJECTS AND STOCHASTIC LINKS

According to the conventional definition, the *dual object* \widehat{G} to a (topological) group G is the set of equivalence classes of irreducible unitary representations of G .

For a finite or compact group, all irreducible representations have finite dimension and the dual object can be identified with the set of irreducible characters.

Let G be a compact group. For $\pi \in \widehat{G}$, denote the dimension of π by $\text{Dim } \pi$. Given a closed subgroup $H \subset G$ and $\rho \in \widehat{H}$, denote by $\text{Dim}(\rho, \pi)$ the multiplicity of ρ in the decomposition of $\pi|_H$. Counting the dimensions we get the identity

$$\text{Dim } \pi = \sum_{\rho \in \widehat{H}} \text{Dim } \rho \text{Dim}(\rho, \pi).$$

Let us form the matrix Λ_H^G whose rows are indexed by elements $\pi \in \widehat{G}$, the columns are indexed by the elements $\rho \in \widehat{H}$, and the entries are given by

$$\Lambda_H^G(\pi, \rho) = \frac{\text{Dim } \rho \text{Dim}(\rho, \pi)}{\text{Dim } \pi}.$$

In other words, $\Lambda_H^G(\pi, \rho)$ is the relative dimension of the ρ -isotypic component in the decomposition of $\pi|_H$.

Evidently, the matrix entries are nonnegative numbers, and (by virtue of the above identity) all row sums are equal to 1, so that Λ_H^G is a *stochastic matrix*. We call it a *stochastic link* and denote by the dashed arrow, $\Lambda_H^G : \widehat{G} \dashrightarrow \widehat{H}$. Informally, we regard Λ_H^G as a “generalized map” from \widehat{G} to \widehat{H} , dual to the inclusion map $H \rightarrow G$.

Let us return to the unitary groups $U(N)$, which are the model example of compact Lie groups. As is well known, the irreducible characters of $U(N)$, viewed as symmetric functions in the matrix eigenvalues u_1, \dots, u_N , have the form

$$\chi_\nu(u_1, \dots, u_N) = \frac{\det \left[u_i^{\nu_j + N - j} \right]_{i,j=1}^N}{\prod_{i,j=1}^N (u_i - u_j)},$$

where the subscript ν is an N -tuple of integers $\nu_1 \geq \dots \geq \nu_N$ called a *signature* of length N (Weyl [55], Zhelobenko [57]). Thus, the dual object $\widehat{U(N)}$ is in one-to-one correspondence with the set $\mathbb{S}_N \subset \mathbb{Z}^N$ formed by the signatures of length N .

Two signatures $\nu \in \mathbb{S}_N$ and $\lambda \in \mathbb{S}_{N-1}$ are said to be *interlaced* if their coordinates satisfy the inequalities $\nu_i \geq \lambda_i \geq \nu_{i+1}$ for every $i = 1, \dots, N-1$; then we write $\lambda \prec \nu$.

Let π_ν denote the irreducible representation with character χ_ν . The classic *Gelfand-Tsetlin branching rule* (Gelfand-Tsetlin [22], Zhelobenko [57]) says that

$$\pi_\nu|_{U(N-1)} = \bigoplus_{\lambda: \lambda \prec \nu} \pi_\lambda,$$

which is equivalent to the character relation

$$\chi_\nu(u_1, \dots, u_{N-1}, 1) = \sum_{\lambda: \lambda \prec \nu} \chi_\lambda(u_1, \dots, u_{N-1}).$$

Recall that the dimension of π_ν , which we denote by $\text{Dim}_N \nu$, is given by the well-known *Weyl's dimension formula*

$$\text{Dim}_N \nu = \prod_{1 \leq i < j \leq N} \frac{\nu_i - \nu_j - i + j}{j - i}.$$

It follows that the stochastic link $\widehat{U(N)} \dashrightarrow \widehat{U(N-1)}$ has the following form

$$\Lambda_{N-1}^N(\nu, \lambda) = \begin{cases} \frac{\text{Dim}_{N-1} \lambda}{\text{Dim}_N \nu}, & \text{if } \lambda \prec \nu, \\ 0, & \text{otherwise.} \end{cases}$$

We explain now how we understand the dual object to the group $U(\infty)$. This group is *wild* (= not type I, see Kirillov [29, Section 8.4]), so the conventional definition of the dual object is inappropriate as it leads to a huge pathological space. For the purpose of the present work it is reasonable to adopt the following definition, which can be formulated in a few different but equivalent ways. Namely, the dual object $\widehat{U(\infty)}$ is:

Version 1. The set of quasi-equivalence classes of finite factor representations of $U(\infty)$.

This formulation follows the approach of Thoma [51] and Voiculescu [54]. Finite factor representations are uniquely (within quasi-equivalence) determined by their normalized traces, which can be characterized as indecomposable positive definite class functions $\chi : U(\infty) \rightarrow \mathbb{C}$ normalized by $\chi(e) = 1$.

Version 2. The set of functions $\chi : U(\infty) \rightarrow \mathbb{C}$ which can be approximated by the *normalized irreducible characters*

$$\tilde{\chi}_\nu := \frac{\chi_\nu}{\chi_\nu(e)} = \frac{\chi_\nu}{\text{Dim}_N \nu},$$

where we assume that $\nu \in \mathbb{S}_N$ varies together with N as N goes to infinity.

The idea of this approach is due to Vershik and Kerov [52], [53]. For more detail, see Okounkov-Olshanski [40].

Version 3. The set of positive definite class functions $\chi : U(\infty) \rightarrow \mathbb{C}$ such that $\chi(e) = 1$ and for arbitrary $g, h \in U(\infty)$ one has

$$\lim_{N \rightarrow \infty} \int_{k \in U(N)} \chi(gkhk^{-1}) dk = \chi(g)\chi(h),$$

where integration is taken with respect to the normalized Haar measure on $U(N)$.

For more detail, see Olshanski [41]. The above limit relation is an analog of the classic *functional equation* for the normalized irreducible characters of compact groups.

Version 4. The categorical projective limit of the sequence of stochastic links

$$\widehat{U(1)} \leftarrow \widehat{U(2)} \leftarrow \widehat{U(3)} \leftarrow \dots$$

For more detail, see Borodin-Olshanski [12], [13], Olshanski [46].

As seen from the third version above, $\widehat{U(\infty)}$ can be identified with a set of positive definite class functions on $U(\infty)$. These functions are called the *indecomposable* or *extreme characters* of $U(\infty)$. Here is their precise description.

First, notice that every element of the group $U(\infty)$ is represented by an infinite unitary matrix $g = [g_{ij}]_{i,j=1}^\infty$ such that $g_{ij} = \delta_{ij}$ when i or j is large enough. Write u_1, u_2, \dots for the eigenvalues of g ; these numbers lie on the unit circle \mathbb{T} and only finitely many of them are distinct from 1. Any class function χ on $U(\infty)$ depends on the eigenvalues only, and we write it as $\chi(u_1, u_2, \dots)$.

Next, we need to introduce some notation. Let $\mathbb{R}_+ \subset \mathbb{R}$ denote the set of non-negative real numbers, \mathbb{R}_+^∞ denote the product of countably many copies of \mathbb{R}_+ , and set

$$\mathbb{R}_+^{4\infty+2} = \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+ \times \mathbb{R}_+.$$

Let $\Omega \subset \mathbb{R}_+^{4\infty+2}$ be the subset of sextuples

$$\omega = (\alpha^+, \beta^+; \alpha^-, \beta^-; \delta^+, \delta^-)$$

such that

$$\begin{aligned} \alpha^\pm &= (\alpha_1^\pm \geq \alpha_2^\pm \geq \dots \geq 0) \in \mathbb{R}_+^\infty, \quad \beta^\pm = (\beta_1^\pm \geq \beta_2^\pm \geq \dots \geq 0) \in \mathbb{R}_+^\infty, \\ \sum_{i=1}^\infty (\alpha_i^\pm + \beta_i^\pm) &\leq \delta^\pm, \quad \beta_1^+ + \beta_1^- \leq 1. \end{aligned}$$

Equip $\mathbb{R}_+^{4\infty+2}$ with the product topology. An important fact is that, in the induced topology, Ω is a locally compact space.

Set

$$\gamma^\pm = \delta^\pm - \sum_{i=1}^\infty (\alpha_i^\pm + \beta_i^\pm)$$

and note that γ^+, γ^- are nonnegative. For $u \in \mathbb{C}^*$ and $\omega \in \Omega$ set

$$\Phi(u; \omega) = e^{\gamma^+(u-1) + \gamma^-(u^{-1}-1)} \prod_{i=1}^\infty \frac{(1 + \beta_i^+(u-1))(1 + \beta_i^-(u^{-1}-1))}{(1 - \alpha_i^+(u-1))(1 - \alpha_i^-(u^{-1}-1))}. \quad (2.1)$$

For any fixed ω , this is a meromorphic function in variable $u \in \mathbb{C}^*$ with possible poles on $(0, 1) \cup (1, +\infty)$. The poles do not accumulate to 1, so that the function is holomorphic in a neighborhood of \mathbb{T} .

Theorem 2.1. *The dual object $\widehat{U(\infty)}$ as defined above can be identified with the space Ω . More precisely, the extreme characters of the group $U(\infty)$ are the functions*

$$\chi_\omega(u_1, u_2, \dots) := \prod_{k=1}^{\infty} \Phi(u_k; \omega),$$

where ω ranges over Ω .

This is a deep result with a long history. See Voiculescu [54] and many references in Borodin-Olshanski [11, Section 1.1].

3. RELATIVE DIMENSION IN THE GELFAND-TSETLIN GRAPH

The *Gelfand-Tsetlin graph* has the vertex set $\mathbb{S}_1 \sqcup \mathbb{S}_2 \sqcup \dots$ consisting of all signatures, and the edges formed by the couples (λ, ν) such that $\lambda \prec \nu$. This is a graded graph with the N th level formed by \mathbb{S}_N .

By a *path* between two vertices $\varkappa \in \mathbb{S}_K$ and $\nu \in \mathbb{S}_N$, $K < N$, we mean a sequence

$$\varkappa = \lambda^{(K)} \prec \lambda^{(K+1)} \prec \dots \prec \lambda^{(N)} = \nu \in \mathbb{S}_N.$$

Such a path can be viewed as an array of numbers

$$\{\lambda_i^{(j)}\}, \quad K \leq j \leq N, \quad 1 \leq i \leq j,$$

satisfying the inequalities $\lambda_i^{(j+1)} \geq \lambda_i^{(j)} \geq \lambda_{i+1}^{(j+1)}$. It is called a *Gelfand-Tsetlin scheme*. If $K = 1$, the scheme has triangular form and if $K > 1$, it has trapezoidal form.

The triangular schemes with a fixed top level $\lambda^{(N)} = \nu$ parameterize the vectors of the *Gelfand-Tsetlin basis* in $\pi_\nu \in \widehat{U(N)}$, see Gelfand-Tsetlin [22], Zhelobenko [57]. The number of such schemes is equal to $\text{Dim}_N \nu$.

The number of paths between \varkappa and ν (or trapezoidal schemes with top ν and bottom \varkappa) will be denoted by $\text{Dim}_{K,N}(\varkappa, \nu)$. It is equal to the quantity $\text{Dim}(\pi_\varkappa, \pi_\nu)$ introduced in the preceding section.

Both $\text{Dim}_N \nu$ and $\text{Dim}_{K,N}(\varkappa, \nu)$ count the lattice points in some bounded convex polytopes.

Adding to the vertex set an additional 0th level formed by a singleton \emptyset , which is joined by edges with all vertices of level 1, one may write $\text{Dim}_N \nu$ as $\text{Dim}_{0,N}(\emptyset, \nu)$.

Note that the matrix Λ_K^N of format $\mathbb{S}_N \times \mathbb{S}_K$ that represents the link $\widehat{U(N)} \dashrightarrow \widehat{U(K)}$ coincides with the matrix product $\Lambda_{N-1}^N \dots \Lambda_K^{K+1}$, and its entries are given by

$$\Lambda_K^N(\nu, \varkappa) = \frac{\text{Dim}_K \varkappa \text{Dim}_{K,N}(\varkappa, \nu)}{\text{Dim}_N \nu}.$$

A sequence of vertices $\{\lambda^{(N)} \in \mathbb{S}_N\}$ is said to be a *regular escape to infinity* if for every fixed vertex $\varkappa \in \mathbb{S}_K$ there exists a limit $\lim_{N \rightarrow \infty} \Lambda_K^N(\lambda^{(N)}, \varkappa)$, and two regular escapes are called *equivalent* if the corresponding limits coincide for every \varkappa . The

set of equivalence classes of regular escapes to infinity is called the *boundary* of the Gelfand-Tsetlin graph and denoted by $\partial\mathbb{GT}$. This is nothing else than one more, this time combinatorial, interpretation of the dual object $\widehat{U(\infty)}$.

Likewise, one can define the boundary $\partial\mathbb{Y}$ of the *Young graph*. That graph encodes the Young branching rule of the symmetric group characters, and $\partial\mathbb{Y}$ parameterizes the extreme characters of the infinite symmetric group.

In the symmetric group case, the stochastic links have the form

$$\Lambda_m^l(\lambda, \mu) = \begin{cases} \frac{\dim \mu \dim \lambda / \mu}{\dim \lambda}, & \text{if } \mu \subset \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

where λ and μ are Young diagrams with l and m boxes, respectively ($l > m$), and $\dim(\cdot)$ denotes the number of standard Young tableaux of a given (possibly skew) shape.

As shown in Okounkov-Olshanski [39],

$$l(l-1)\dots(l-m+1)\frac{\dim \lambda / \mu}{\dim \lambda} = s_\mu^*(\lambda_1, \lambda_2, \dots), \quad (3.1)$$

where s_μ^* is the so-called *shifted Schur function*. Informally, the meaning of this result is that the quantity in the left-hand side behaves as a “regular” function in variable λ . Formula (3.1) is well suited for asymptotic analysis and makes it possible to quickly find the boundary $\partial\mathbb{Y}$, thus obtaining a proof of Thoma’s theorem about the characters of the infinite symmetric group (Thoma [51]), see Kerov-Okounkov-Olshanski [26] and Borodin-Olshanski [13, Section 3.3].

By analogy, one can ask what can be said about the function

$$F_\varkappa(\nu) := \frac{\text{Dim}_{K,N}(\varkappa, \nu)}{\text{Dim}_N \nu}.$$

This problem was investigated in our recent paper Borodin-Olshanski [11]. To give a flavor of what we get, I will formulate one of the results in the simplest (but nontrivial!) case when $K = 1$.

Theorem 3.1. *Let $\varkappa = k$ range over $\mathbb{S}_1 = \mathbb{Z}$, ν range over \mathbb{S}_N , and write $F_k(\nu)$ instead of $F_\varkappa(\nu)$. Set*

$$H^*(t; \nu) = \prod_{j=1}^N \frac{t+j}{t+j-\nu_j},$$

where t is a formal variable.

Then the following identity holds

$$H^*(t; \nu) = \sum_{k \in \mathbb{Z}} F_k(\nu) \frac{(t+1)\dots(t+N)}{(t+1-k)\dots(t+N-k)}. \quad (3.2)$$

This is a kind of generating series for $F_k(\nu)$ from which one can extract a contour integral representation for $F_k(\nu)$.

Let $\varphi_n(\omega)$ denote the coefficients of the Laurent expansion of the function $u \mapsto \Phi(u; \omega)$ on \mathbb{T} :

$$\Phi(u; \omega) = \sum_{n \in \mathbb{Z}} \varphi_n(\omega) u^n. \quad (3.3)$$

The identity (3.2) mimics the Laurent expansion (3.3), and in a limit transition, (3.2) turns into (3.3).

I briefly list further results of [11].

There is an extension of (3.2) to arbitrary $K = 1, 2, \dots$ and $\varkappa \in \mathbb{S}_K$:

$$\prod_{i=1}^K H^*(t_i; \nu) = \sum_{\varkappa \in \mathbb{GT}_K} F_{\varkappa}(\nu) \mathfrak{S}_{\varkappa|N}(t_1, \dots, t_K), \quad (3.4)$$

where $\mathfrak{S}_{\varkappa|N}(t_1, \dots, t_K)$ is a certain ‘‘Schur-type’’ rational symmetric function in variables t_1, \dots, t_K :

$$\mathfrak{S}_{\varkappa|N}(t_1, \dots, t_K) = \text{const} \frac{\det[G_{\varkappa_j + K - j|N}(t_i)]_{i,j=1}^N}{\prod_{1 \leq i < j \leq N} (t_i - t_j)}$$

(here $G_{k|N}(t)$ are certain univariate rational functions).

We show that $F_{\varkappa}(\nu)$ also has a similarity with the Schur function. Namely, there is an analog of the Jacobi-Trudi formula:

$$F_{\varkappa}(\nu) = \det[F_{\varkappa_i - i + j}^{(j)}(\nu)]_{i,j=1}^K,$$

where $F_k^{(j)}(\nu)$, $k \in \mathbb{Z}$, is a certain modification of $F_k(\nu)$. Note that a similar modified Jacobi-Trudi identity holds for the shifted Schur functions (Okounkov-Olshanski [39]) as well as for other analogs of Schur functions (Macdonald [35], Nakagawa-Noumi-Shirakawa-Yamada [36], Sergeev-Veselov [50]).

As both functions on the right-hand side of (3.4) are similar to the Schur functions, this relation may be viewed as a kind of the Cauchy identity.

From (3.2) one can derive a closed formula for $F_k(\nu)$ (in the form of a contour integral representation), and the same can be done for the modified functions $F_k^{(j)}(\nu)$. Like (3.1), the resulting formula is well adapted to asymptotic analysis, which enables us to re-derive Theorem 2.1 in a way very similar to that used in Kerov-Okounkov-Olshanski [26] for the infinite symmetric group $S(\infty)$.

Note that Petrov [49] found a different approach to the results of [11] together with a q -version of them.

Finally note that the results of [11] can also be extended to symplectic and orthogonal groups (work in progress).

4. THE ZW-MEASURES

Let the symbol $\mathcal{P}(X)$ denote the set of probability measures on a space X . Given a measure $M \in \mathcal{P}(\mathbb{S}_N)$ with weights $M(\nu)$, its composition with the link Λ_{N-1}^N is a measure $M\Lambda_{N-1}^N \in \mathcal{P}(\mathbb{S}_{N-1})$ defined by

$$(M\Lambda_{N-1}^N)(\lambda) = \sum_{\nu \in \mathbb{S}_N} M(\nu) \Lambda_{N-1}^N(\nu, \lambda), \quad \lambda \in \mathbb{S}_{N-1}.$$

A family of measures $\{M_N \in \mathcal{P}(\mathbb{S}_N) : N = 1, 2, \dots\}$ is said to be *coherent* if the measures are consistent with the links in the sense that $M_N \Lambda_{N-1}^N = M_{N-1}$ for every $N \geq 2$.

For $\omega \in \Omega$ and $\nu \in \mathbb{S}_N$ we denote by $\Lambda_N^\infty(\omega, \nu)$ the coefficients in the expansion of the N -fold product $\Phi(u_1; \omega) \dots \Phi(u_N; \omega)$ on the normalized irreducible characters:

$$\Phi(u_1; \omega) \dots \Phi(u_N; \omega) = \sum_{\nu \in \mathbb{S}_N} \Lambda_N^\infty(\omega, \nu) \tilde{\chi}_\nu(u_1, \dots, u_N) = \sum_{\nu \in \mathbb{S}_N} \Lambda_N^\infty(\omega, \nu) \frac{\chi_\nu(u_1, \dots, u_N)}{\text{Dim}_N \nu}.$$

One readily shows that

$$\Lambda_N^\infty(\omega, \nu) = \text{Dim}_N \nu \cdot \det[\varphi_{\nu_i - i + j}(\omega)]_{i,j=1}^N. \quad (4.1)$$

Note that Λ_N^∞ is a Markov kernel meaning that for ω fixed, $\Lambda_N^\infty(\omega, \cdot)$ is a probability measure on \mathbb{S}_N . We regard Λ_N^∞ as a “link” $\Omega \dashrightarrow \mathbb{S}_N$.

There is a natural one-to-one correspondence $\{M_N\} \leftrightarrow M_\infty$ between the coherent families $\{M_N\}$ and the measures $M_\infty \in \mathcal{P}(\Omega)$ given by

$$M_N = M_\infty \Lambda_N^\infty, \quad N = 1, 2, 3, \dots$$

Let us say that a quadruple z, z', w, w' of complex parameters is *admissible* if the following conditions hold: firstly, for every integer k , one has $(z+k)(z'+k) > 0$ and $(w+k)(w'+k) > 0$; secondly, $\Re(z+z'+w+w') > -1$. As readily seen, the first condition is equivalent to saying that each of pairs (z, z') and (w, w') belongs to the subset $\mathcal{Z} \subset \mathbb{C}^2$ defined by

$$\begin{aligned} \mathcal{Z} := & \{(\zeta, \zeta') \in (\mathbb{C} \setminus \mathbb{Z})^2 \mid \zeta' = \bar{\zeta}\} \\ & \cup \{(\zeta, \zeta') \in (\mathbb{R} \setminus \mathbb{Z})^2 \mid m < \zeta, \zeta' < m+1 \text{ for some } m \in \mathbb{Z}\}. \end{aligned} \quad (4.2)$$

For $N = 1, 2, \dots$ and $\nu \in \mathbb{S}_N$ set

$$\begin{aligned} M'_{z,z',w,w'|N}(\nu) = & \prod_{i=1}^N \left(\frac{1}{\Gamma(z - \nu_i + i) \Gamma(z' - \nu_i + i)} \right. \\ & \times \left. \frac{1}{\Gamma(w + N + 1 + \nu_i - i) \Gamma(w' + N + 1 + \nu_i - i)} \right) \cdot (\text{Dim}_N \nu)^2. \end{aligned}$$

If (z, z', w, w') is admissible, then $M'_{z,z',w,w'|N}(\nu) > 0$, the series $\sum_{\nu \in \mathbb{S}_N} M'_{z,z',w,w'|N}(\nu)$ is convergent, and its sum equals

$$C_{z,z',w,w'|N} := \prod_{i=1}^N \frac{\Gamma(z + z' + w + w' + i)}{\Gamma(z + w + i)\Gamma(z + w' + i)\Gamma(z' + w + i)\Gamma(z' + w' + i)\Gamma(i)}.$$

This is a multivariate version of Dougall's formula (1.3) we started with.

Consequently, the quantities

$$M_{z,z',w,w'|N}(\nu) := M'_{z,z',w,w'|N}(\nu) / C_{z,z',w,w'|N}, \quad \nu \in \mathbb{S}_N,$$

determine a probability measure on \mathbb{S}_N . For $N = 1$ this measure coincides with the measure $M_{z,z',w,w'}$ on \mathbb{Z} introduced in the very beginning.

The measures $M_{z,z',w,w'|N}$ are a special case of the *orthogonal polynomial ensembles* (about this notion see König [30]).

Namely, let us associate with $\nu \in \mathbb{S}_N$ a collection (n_1, \dots, n_N) of pairwise distinct integers by setting

$$n_i := \nu_i + N - i, \quad i = 1, \dots, N. \quad (4.3)$$

Under the correspondence $\nu \mapsto (n_1, \dots, n_N)$, the measure $M_{z,z',w,w'|N}$ determines an ensemble of random N -point configurations on \mathbb{Z} , and it is the orthogonal polynomial ensemble related to the family of polynomials orthogonal with respect to weight $M_{z+N-1,z'+N-1,w,w'}$. These curious orthogonal polynomials were discovered by Askey [1] and later investigated by Lesky [31], [32]. They are relatives of the classical Hahn polynomials. For more detail, see Borodin-Olshanski [12].

Theorem 4.1. *Fix a quadruple (z, z', w, w') of admissible parameters and let N range over $\{1, 2, \dots\}$. The family $\{M_{z,z',w,w'|N}(\nu)\}$ just defined is a coherent family.*

Different proofs are given in Olshanski [42], [43]. The latter paper actually contains a more general result (the links and the measures depend on an additional parameter — the “Jack parameter”). In Olshanski-Osinenko [48], the theorem is extended to other branching graphs including those related to the orthogonal and symplectic groups.

Corollary 4.2. *For every admissible quadruple (z, z', w, w') there exists a probability measure $M_{z,z',w,w'|\infty}$ on the space Ω , uniquely determined by the property that*

$$M_{z,z',w,w'|\infty} \Lambda_N^\infty = M_{z,z',w,w'|N}, \quad N = 1, 2, \dots$$

Both $M_{z,z',w,w'|N}$ and $M_{z,z',w,w'|\infty}$ are called the *zw-measures*. They are analogs of the *z-measures* which arise in the context of the symmetric groups (see Borodin-Olshanski [8], the recent survey Olshanski [44], and also Section 7 below). A common feature of all these measures is that they serve as the laws of *determinantal point processes* (about those, see Borodin [4] and references therein).

It is worth noting that the zw-measures on \mathbb{S}_N are introduced by an explicit formula while our definition of the zw-measures on Ω is indirect: Corollary 4.2 only provides the explicit values for the expectation of certain functionals.

Our interest in the zw-measures on Ω is motivated by the fact that they arise in the problem of harmonic analysis on the infinite-dimensional unitary group (Olshanski [42], Borodin-Olshanski [9], [10]).

5. MARKOV PROCESSES

We need a few basic notions from the theory of Markov processes (see Liggett [33], Ethier-Kurtz [21]).

Let X be a locally compact space and $C_0(X)$ denote the space of real-valued continuous functions on X vanishing at infinity; $C_0(X)$ is a Banach space with respect to the supremum norm. A *Feller semigroup* on X is a strongly continuous semigroup $P(t)$, $t \geq 0$, of operators on $C_0(X)$ which are given by Markov kernels. This means that

$$(P(t)f) = \int_X P(t; x, dy) f(y), \quad x \in X, \quad f \in C(X),$$

where $P(t; x, \cdot) \in \mathcal{P}(X)$ for every $t \geq 0$ and $x \in X$. A well-known abstract theorem says that a Feller semigroup gives rise to a Markov process on X with transition function $P(t; x, dy)$. The processes derived from Feller semigroups are called *Feller processes*; they form a particularly nice subclass of general Markov processes.

A Feller semigroup $P(t)$ is uniquely determined by its *infinitesimal generator*. This is a (typically, unbounded) closed operator A on $C_0(X)$ given by

$$Af = \lim_{t \rightarrow +0} \frac{P(t)f - f}{t}.$$

The *domain* of A , denoted by $\text{Dom } A$, is the (algebraic) subspace formed by those functions $f \in C_0(X)$ for which the above limit exists; $\text{Dom } A$ is always a dense subspace. A *core* of A is a subspace $\mathcal{F} \subset \text{Dom } A$ such that A is the closure of $A|_{\mathcal{F}}$. One can say that a core is an “essential domain” for A . The full domain is often difficult to describe explicitly, and then one is satisfied by exhibiting a core with the action of A on it.

The Markov chain on $X = \mathbb{Z}$ mentioned in Section 1 is an example of a Feller process. Now we are going to define its multidimensional analog with $X = \mathbb{S}_N$.

First we need to introduce some notation. It is convenient to use the correspondence (4.3) to pass from \mathbb{S}_N to the subset $\Omega_N \subset \mathbb{Z}^N$ formed by the N -tuples $\tilde{n} = (n_1 > \cdots > n_N)$. Let

$$V(\tilde{n}) := \prod_{1 \leq i < j \leq N} (n_i - n_j)$$

and ε_k denote the k th basis vector in $\mathbb{Z}^N \subset \mathbb{R}^N$.

We introduce a partial difference operator $D_{z,z',w,w'|N}$ on Ω_N depending on an admissible quadruple (z, z', w, w') , as follows

$$\begin{aligned} (D_{z,z',w,w'|N}f)(\tilde{n}) &= \sum_{k=1}^N \left(\frac{V(\tilde{n} + \varepsilon_k)}{V(\tilde{n})} (z + N - 1 - n_k)(z' + N - 1 - n_k)(f(\tilde{n} + \varepsilon_k) - f(\tilde{n})) \right. \\ &\quad \left. + \frac{V(\tilde{n} - \varepsilon_k)}{V(\tilde{n})} (w + n_k)(w' + n_k)(f(\tilde{n} - \varepsilon_k) - f(\tilde{n})) \right) + \text{const}, \end{aligned} \quad (5.1)$$

where the constant term is chosen so that the operator annihilates the constant functions.

This difference operator is well defined on Ω_N , because if $\tilde{n} + \varepsilon_k \notin \Omega_N$, or $\tilde{n} - \varepsilon_k \notin \Omega_N$, then $V(\tilde{n} + \varepsilon_k)$ or, respectively, $V(\tilde{n} - \varepsilon_k)$ vanishes.

In the case $N = 1$ the operator reduces to the ordinary difference operator $D_{z,z',w,w'}$ defined in (1.4).

Theorem 5.1. *Let (z, z', w, w') be an admissible quadruple of parameters. For every $N = 1, 2, \dots$ there exists a Feller semigroup on $\Omega_N \subset \mathbb{Z}^N$ whose generator is given by the partial difference operator (5.1) with domain*

$$\{f \in C_0(\Omega_N) : D_{z,z',w,w'|N}f \in C_0(\Omega_N)\}.$$

See Borodin-Olshanski [12, Section 5].

As pointed out in [12, Subsection 1.3], the Feller process provided by Theorem 5.1 can be viewed as the *Doob h -transform* of a collection of N independent Markov chains on \mathbb{Z} , with h equal to the Vandermonde $V(\tilde{n})$.

Using the bijection (4.3) between \mathbb{S}_N and Ω_N we may interpret the semigroup of Theorem 5.1 as a Feller semigroup on $C_0(\mathbb{S}_N)$. Let us denote it by $P_{z,z',w,w'|N}(t)$.

Theorem 5.2. *The measure $M_{z,z',w,w'|N}$ on \mathbb{S}_N serves as the stationary distribution for the Feller process defined by the semigroup $P_{z,z',w,w'|N}(t)$.*

See Borodin-Olshanski [12, Section 7].

Theorem 5.3. *Let (z, z', w, w') be a fixed admissible quadruple and N range over $\{1, 2, \dots\}$. The links Λ_{N-1}^N intertwine the semigroups $P_{z,z',w,w'|N}(t)$.*

See Borodin-Olshanski [12, Section 6]. Let us comment on this result. The link Λ_{N-1}^N determines an operator $f \mapsto \Lambda_{N-1}^N f$ transforming bounded functions on \mathbb{S}_{N-1} into bounded functions on \mathbb{S}_N by

$$(\Lambda_{N-1}^N f)(\nu) = \sum_{\lambda \in \mathbb{S}_{N-1}} \Lambda_{N-1}^N(\nu, \lambda) f(\lambda).$$

One proves that Λ_{N-1}^N is “Feller” in the sense that it maps $C_0(\mathbb{S}_{N-1})$ into $C_0(\mathbb{S}_N)$, and the claim of the theorem means that the following commutativity relations hold

$$P_{z,z',w,w'|N}(t) \Lambda_{N-1}^N = \Lambda_{N-1}^N P_{z,z',w,w'|N-1}(t), \quad N = 2, 3, \dots, \quad t \geq 0.$$

One also proves the Feller property for the link Λ_N^∞ meaning that Λ_N^∞ maps $C_0(\mathbb{S}_N)$ into $C_0(\Omega)$. (Because of (4.1), this amounts to the fact that the functions $\varphi_n(\omega)$ lie in $C_0(\Omega)$.) Then, using a very simple argument, one derives from the above theorems the following result:

Theorem 5.4. (i) *For every admissible quadruple (z, z', w, w') , there exists a unique Feller semigroup $P_{z,z',w,w'|\infty}(t)$ on $C_0(\Omega)$ such that*

$$P_{z,z',w,w'|\infty}(t)\Lambda_N^\infty = \Lambda_N^\infty P_{z,z',w,w'|N}(t), \quad N = 1, 2, 3, \dots, \quad t \geq 0.$$

(ii) *The measure $M_{z,z',w,w'|\infty}$ on Ω serves as the stationary distribution for the corresponding Feller process.*

This is one of the main results of Borodin-Olshanski [12] (see also the expository paper Olshanski [46]).

6. THE REPRESENTATION RING AND THE GENERATOR

In this section I briefly review the recent results from my paper [47].

Let R^S denote the graded representation ring of all symmetric groups $S(n)$ collected together, with the multiplication determined by the operation of induction $\text{Ind}_{S(m) \times S(n)}^{S(m+n)}$ from Young subgroups: see Macdonald [34, Chapter I, Section 7]. As is clearly shown there, the original Frobenius' approach to the classification of the symmetric group characters essentially relies on the canonical isomorphism between R^S and the ring of symmetric functions (see also Zelevinsky [56]).

One can ask if there is a reasonable analog of the ring R^S for the unitary groups (as well as for other families of classical compact groups). The answer is yes, but it is necessary to take into account the fact that the operation of induction $\text{Ind}_{U(M) \times U(N)}^{U(M+N)}$ leads to *infinite* sums of irreducible representations.

Let $\mathbb{C}[[\dots, \varphi_{-1}, \varphi_0, \varphi_1, \dots]]$ be the \mathbb{C} -algebra of formal power series in countably many indeterminates φ_n , $n \in \mathbb{Z}$, and let

$$\mathbb{C}[[\dots, \varphi_{-1}, \varphi_0, \varphi_1, \dots]]_{\text{bounded}} \subset \mathbb{C}[[\dots, \varphi_{-1}, \varphi_0, \varphi_1, \dots]]$$

be the subalgebra of series with bounded degree. This subalgebra is a graded algebra: its N th homogeneous component is formed by the homogeneous series of degree N .

According to our definition, the representation ring for the family of unitary groups, denoted by R^U , can be identified with $\mathbb{C}[[\dots, \varphi_{-1}, \varphi_0, \varphi_1, \dots]]_{\text{bounded}}$.

The algebra R^U contains all the irreducible characters χ_ν (where $\nu \in \mathbb{S}_N$, $N = 1, 2, \dots$): namely we identify

$$\chi_\nu = \det[\varphi_{\nu_i - i + j}]_{i,j=1}^N. \quad (6.1)$$

We introduce an “adic” topology in R^U . With respect to it, both the monomials in the indeterminates φ_n and the characters χ_ν (together with the unity element 1) are “topological bases”.

Now let us fix an arbitrary quadruple (z, z', w, w') of complex parameters and introduce the following formal differential operator in countably many indeterminates $\{\varphi_n : n \in \mathbb{Z}\}$

$$\mathbb{D}_{z,z',w,w'} = \sum_{n \in \mathbb{Z}} A_{nn} \frac{\partial^2}{\partial \varphi_n^2} + 2 \sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ n_1 > n_2}} A_{n_1 n_2} \frac{\partial^2}{\partial \varphi_{n_1} \partial \varphi_{n_2}} + \sum_{n \in \mathbb{Z}} B_n \frac{\partial}{\partial \varphi_n},$$

where, for any indices $n_1 \geq n_2$,

$$\begin{aligned} A_{n_1 n_2} &= \sum_{p=0}^{\infty} (n_1 - n_2 + 2p + 1) (\varphi_{n_1+p+1} \varphi_{n_2-p} + \varphi_{n_1+p} \varphi_{n_2-p-1}) \\ &\quad - (n_1 - n_2) \varphi_{n_1} \varphi_{n_2} - 2 \sum_{p=1}^{\infty} (n_1 - n_2 + 2p) \varphi_{n_1+p} \varphi_{n_2-p} \end{aligned}$$

and, for any $n \in \mathbb{Z}$,

$$\begin{aligned} B_n &= (n + w + 1)(n + w' + 1) \varphi_{n+1} + (n - z - 1)(n - z' - 1) \varphi_{n-1} \\ &\quad - ((n - z)(n - z') + (n + w)(n + w')) \varphi_n. \end{aligned}$$

The operator $\mathbb{D}_{z,z',w,w'}$ correctly defines a linear map $R^U \rightarrow R^U$. Notice that only the coefficients B_n depend on the parameters (z, z', w, w') .

Our aim is to interpret $\mathbb{D}_{z,z',w,w'}$ as an operator acting on a certain linear subspace $\mathcal{F} \subset C_0(\Omega)$.

First, we define \mathcal{F} as the algebraic linear span of all the elements $\chi_\nu \in R^U$ (where $\nu \in \mathbb{S}_N$, $N = 1, 2, \dots$).

Proposition 6.1. *The subspace \mathcal{F} is invariant under the action of operator $\mathbb{D}_{z,z',w,w'}$.*

Next, we embed \mathcal{F} into $C_0(\Omega)$. To this end we identify every formal indeterminate φ_n with the function $\varphi_n(\omega)$ on Ω introduced in (3.3). (We have already mentioned that these functions lie in $C_0(\Omega)$.) Then, by virtue of (6.1), all elements $\chi_\nu \in R^R$ are turned into functions $\chi_\nu(\omega)$ belonging to $C_0(\Omega)$. In this way \mathcal{F} becomes a subspace of $C_0(\Omega)$.

In the next theorem we use the notion of a core defined in the beginning of Section 5.

Theorem 6.2. *Assume (z, z', w, w') is admissible and let $A_{z,z',w,w'}$ denote the generator of the Feller semigroup $P_{z,z',w,w'|\infty}(t)$ from Theorem 5.4.*

The subspace $\mathcal{F} \subset C_0(\Omega)$ is an invariant core for the generator $A_{z,z',w,w'}$, and its action on \mathcal{F} is implemented by the operator $\mathbb{D}_{z,z',w,w'}$.

Our construction of the Feller processes on Ω is rather abstract and indirect, but Theorem 6.2 provides a piece of concrete information about them.

7. THE YOUNG BOUQUET

Here I review the results of Borodin-Olshanski [13].

Consider the infinite chain of finite symmetric groups with natural embeddings

$$S(1) \subset S(2) \subset S(3) \subset \dots \quad (7.1)$$

and let $S(\infty)$ denote the union of all these groups. In other words, $S(\infty)$ is the group of finitary permutations of the set $\{1, 2, 3, \dots\}$. The characters of both the symmetric and unitary groups are related to the Schur symmetric functions. The similarity between the characters of the inductive limit groups $S(\infty)$ and $U(\infty)$ is even more apparent. On the other hand, the groups themselves are structurally very different. We suggest an explanation of this phenomenon.

As we tried to demonstrate in Section 3, the combinatorial base of the character theory of $U(\infty)$ is the Gelfand-Tsetlin graph. Its counterpart in the symmetric group case is the *Young graph*, also called the *Young lattice*. The vertex set of the Young graph is the set of all Young diagrams, and two diagrams are joined by an edge if they differ by a single box. The graph is graded: its n th level ($n = 0, 1, 2, \dots$) consists of the diagrams with n boxes. The Young graph encodes the branching of the irreducible characters of the chain (7.1), just as the Gelfand-Tsetlin graph does for the chain (1.1) (Vershik-Kerov [52]).

The description of the extreme characters of $S(\infty)$ was given by Thoma [51]. It can be reformulated as the description of the boundary of the Young graph. For various proofs of the fundamental Thoma's theorem, see Vershik-Kerov [52], Okounkov [37], Kerov-Okounkov-Olshanski [26].

In [13] we introduce and study a new object which occupies an intermediate position between the Gelfand-Tsetlin graph and the Young graph, and makes it possible to see a clear connection between them. This new object is called the *Young bouquet* and denoted as \mathbb{YB} . It is not an ordinary graph. However, \mathbb{YB} is a graded poset, similarly to the Gelfand-Tsetlin and Young graphs.

One new feature is that the grading in \mathbb{YB} is not discrete but continuous: the grading level is marked by a positive real number. By definition, the elements of \mathbb{YB} of a given level $r > 0$ are pairs (λ, r) , where λ is an arbitrary Young diagram. The partial order in \mathbb{YB} is defined as follows: $(\mu, r) < (\lambda, r')$ if $r < r'$ and diagram μ is contained in diagram λ (or coincides with it).

From the definition of the Young bouquet one sees that it is closely related to the Young graph. We explain in [13] how various notions related to the Young graph are modified in the context of the Young bouquet. In particular, we are led to consider *Young tableaux with continuous entries* as a counterpart of the conventional tableaux.

Let \mathbb{YB}_r stand for the r th level of the poset \mathbb{YB} ; this is simply a copy of the set \mathbb{Y} of all Young diagrams. For every couple of positive reals $r' > r$ we define a link ${}^{\mathbb{YB}}\Lambda_r^{r'} : \mathbb{YB}_{r'} \dashrightarrow \mathbb{YB}_r$, which is a stochastic matrix of format $\mathbb{Y} \times \mathbb{Y}$ that depends

only of the ratio $r' : r$. The links satisfy the compatibility relation

$${}^{\mathbb{YB}}\Lambda_{r'}^{r''} {}^{\mathbb{YB}}\Lambda_r^{r'} = {}^{\mathbb{YB}}\Lambda_r^{r''}, \quad r'' > r' > r > 0,$$

which enables us to define the *boundary of the Young bouquet* in the spirit of the fourth version of the definition given in Section 2.

The boundary of \mathbb{YB} and the boundary of the Young graph are directly connected: the former is the cone whose base is the latter. Namely, the boundary of \mathbb{YB} , called the *Thoma cone*, can be identified with the subset in $\mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \times \mathbb{R}_+$ formed by the triples (α, β, δ) such that

$$\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq 0), \quad \beta = (\beta_1 \geq \beta_2 \geq \cdots \geq 0), \quad \delta \geq 0$$

and

$$\sum_{i=1}^{\infty} (\alpha_i + \beta_i) \leq \delta,$$

while the boundary of the Young graph, called the *Thoma simplex*, can be identified with the section $\delta = 1$ of the Thoma cone.

On the other hand, we explain how the Young bouquet \mathbb{YB} is connected with the Gelfand-Tsetlin graph. We consider the subgraph \mathbb{GT}^+ of the Gelfand-Tsetlin graph spanned by the signatures with nonnegative coordinates. The N th level vertices of \mathbb{GT}^+ can be viewed as pairs (λ, N) , where λ is a Young diagram with at most N nonzero rows.

We show (Theorem 4.4.1 in [13]) that \mathbb{YB} is a *degeneration* of \mathbb{GT}^+ in the following sense.

Theorem 7.1. *Fix arbitrary positive numbers $r' > r > 0$ and arbitrary two Young diagrams λ and μ such that $\mu \subseteq \lambda$. Let two positive integers $N' > N$ go to infinity in such a way that $N'/N \rightarrow r'/r$. Then*

$$\lim \Lambda_N^{N'}((\lambda, N'), (\mu, N)) = {}^{\mathbb{YB}}\Lambda_r^{r'}(\lambda, \mu). \quad (7.2)$$

We also exhibit a limit procedure turning the boundary of \mathbb{GT}^+ (which is a subset of Ω) into the boundary of \mathbb{YB} , the Thoma cone (Theorem 4.5.1 in [13]).

Next, we show that along the degeneration $\mathbb{GT}^+ \rightarrow \mathbb{YB}$, some degenerate versions of zw-measures on the levels of the Gelfand-Tsetlin graph turn into the *z-measures* on the set \mathbb{Y} .

The z-measures on \mathbb{Y} are a distinguished particular case of Okounkov's *Schur measures* (Okounkov [38], Borodin-Okounkov [7]). For the first time, the z-measures appeared in Borodin-Olshanski [8] in connection with the problem of harmonic analysis on the infinite symmetric group stated in Kerov-Olshanski-Vershik[27] (see also the detailed paper [28]).

The z-measures depend on a pair $(z, z') \in \mathcal{Z}$ of parameters (see (4.2)) and the additional parameter $r > 0$ indexing the level of \mathbb{YB} . The measures are consistent with the links ${}^{\mathbb{YB}}\Lambda_r^{r'}$ and give rise to certain probability measures $M_{z, z'|_\infty}$ on the

Thoma cone, in complete similarity with the case of Gelfand-Tsetlin graph (see Corollary 4.2 above).

The parallelism between the Young bouquet and the Gelfand-Tsetlin graph also extends to the theory outlined in Section 5. In Borodin-Olshanski [14] we show that using the same approach, one can construct a family of Feller Markov processes on the Thoma cone.

8. NOTES AND COMPLEMENTS

8.1. In connection with the material of Section 3 see also Olshanski [45].

8.2. There exist other values of parameters (z, z', w, w') for which coherent families $\{M_{z,z',w,w'|N}; N = 1, 2, \dots\}$ are still well defined and give rise to certain probability measures $M_{z,z',w,w'|\infty}$ on the boundary Ω . Only these measures are *degenerate* meaning that the support of $M_{z,z',w,w'|N}$ is a proper subset of \mathbb{S}_N .

For instance, one can take

$$z = m, \quad z' = m + a, \quad w = 0, \quad w' = b,$$

where m is a positive integer, $a > -1$, and $b > -1$. Then the corresponding measure on Ω is concentrated on the subset

$$\{\omega : \delta^+ = \beta_1^+ + \dots + \beta_m^+, \quad 1 \geq \beta_1^+ \geq \dots \geq \beta_m^+ \geq 0, \\ \text{all other coordinates equal } 0\} \subset \Omega$$

and takes the form

$$(\text{normalizing constant}) \cdot \prod_{i=1}^m t_i^b (1 - t_i)^a \cdot \prod_{1 \leq i < j \leq m} (t_i - t_j)^2 \cdot \prod_{i=1}^m dt_i, \quad (8.1)$$

where we use the notation

$$(t_1, \dots, t_m) := (1 - \beta_m^+, \dots, 1 - \beta_1^+).$$

The measure (8.1) is a multidimensional version of the Euler Beta distribution of the type appearing in Selberg's integral, and the coherency property of the degenerate zw-measures is related to a generalized Selberg integral (Olshanski [43, Section 5]).

8.3. In the case of degenerate measures, the construction of Section 6 produces a diffusion Markov process on the m -dimensional simplex

$$\{(t_1, \dots, t_m) : 1 \geq t_1 \geq \dots \geq t_m \geq 0\}$$

whose generator is the m -variate *Jacobi differential operator*

$$\begin{aligned} D_m^{(a,b)} &:= \frac{1}{V_m} \circ \left(\sum_{i=1}^m \left(t_i(1-t_i) \frac{\partial^2}{\partial t_i^2} + [b+1-(a+b+2)t_i] \frac{\partial}{\partial t_i} \right) \right) \circ V_m + \text{const} \\ &= \sum_{i=1}^m \left(t_i(1-t_i) \frac{\partial^2}{\partial t_i^2} + \left[b+1-(a+b+2)t_i + \sum_{j:j \neq i} \frac{2t_i(1-t_i)}{t_i-t_j} \right] \frac{\partial}{\partial t_i} \right), \end{aligned}$$

where V_m denotes the Vandermonde,

$$V_m := \prod_{1 \leq i < j \leq m} (t_i - t_j),$$

and

$$\text{const} = \sum_{k=0}^{m-1} k(k+a+b+1).$$

This fact is substantially exploited in the proof of Theorem 6.2.

The same diffusion process also arises in a different context, see Gorin [23].

8.4. Gorin [25] considered the “ q -Gelfand-Tsetlin graph” (which amounts to introducing a q -deformation of the links $\Lambda_{N-1}^N : \mathbb{S}_N \dashrightarrow \mathbb{S}_{N-1}$) and found the corresponding boundary. Under this deformation, the α -parameters disappear while the β -parameters survive but become discrete.

For the “ q -Gelfand-Tsetlin graph”, analogs of zw-measures and related processes are unknown. However, Borodin and Gorin [6] applied the approach outlined in Section 6 above to constructing Feller processes of a different kind.

8.5. Let \mathcal{T} stand for the space of infinite monotone paths in the Gelfand-Tsetlin graph, which are the same as *infinite Gelfand-Tsetlin schemes*. The path space \mathcal{T} plays an important role in our theory.

There exists a one-to-one correspondence $\mathcal{P}(\Omega) \leftrightarrow \mathcal{P}_{\text{central}}(\mathcal{T})$ between probability measures on Ω and some kind of *Gibbs measures* (or *central measures*, in Vershik-Kerov’s terminology) on \mathcal{T} (Olshanski [42, Proposition 10.3]).

Using this correspondence, Gorin [24] proved that the zw-measures on Ω are pairwise mutually singular.

Via this correspondence, the semigroup $P_{z,z',w,w'|\infty}(t)$ defines an evolution of central measures. It is natural to ask if there exists a Markov process on \mathcal{T} that agrees with that evolution when restricted to the central measures. In Borodin-Olshanski [12] we construct one such process (for every admissible (z, z', w, w')).

In Borodin-Olshanski [15] we present arguments in favor of the conjecture that a similar construction can be carried out for the Young bouquet. (Then the path space consists of infinite Young tableaux with continuous entries.) The conjectural process should be *piecewise deterministic* meaning that it is a combination of a dynamical system with a jump Markov process.

8.6. Note that in the literature one can find a number of examples of “Markov intertwiners” (that is, Markov kernels intertwining two Markov processes); see, e.g. Biane [2], [3], Dubédat [18], Carmona-Petit-Yor[16]. However, the use of Markov intertwiners for constructing infinite-dimensional Markov processes seems to be new.

8.7. In the ICM lecture [5], Borodin demonstrates how intertwined Markov processes of the type considered above are applied to analyzing the large time behavior of certain interacting particle systems and random growth models.

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